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LETTER TO THE EDITOR

A fractal model for the low-field Hall effect at three-dimensional percolation threshold

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Abstract. A fractal model is proposed to determine the critical behaviour of the low-field Hall effect in a three-dimensional metal-non-metal composite at the percolation threshold. The fractal lattice is constructed to imitate the geometric texture of three-dimensional percolation clusters at criticality and to realise the Hall problem on a three-dimensional discrete lattice. The exponents describing the power law dependence on scale length of the Hall and ohmic conductivities are found.

Recently, there has been increasing interest in exact mathematical fractals. The main reason is that solution of many important equations of physics on these lattices adds to our understanding of the geometric and topological properties that are relevant to modelling the corresponding physical processes (Mandelbrot 1982, Vicsek 1983, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983, Blumenfeld and Aharony 1985, Martin and Keefer 1985). The percolating infinite cluster is one of the most intensively studied random fractals (Deutscher *et al* 1983, Stauffer 1979, 1985, Stanley and Coniglio 1983, Kirkpatrick 1979, Kapitulnik and Deutscher 1984). Various geometrical models have been proposed to imitate the infinite incipient cluster at the percolation threshold and it has been of great interest to understand the effects of these different geometries on the transport properties near the percolation threshold (Mandelbrot 1984a,b, Mandelbrot and Given 1984, Nagatani 1985a,b).

In the past the Hall effect has been used extensively to investigate the metal-non-metal transition in a variety of disordered systems. An effective-medium theory (EMT) and a simulation approach have been used to discuss the properties of the Hall effect in conductors with macroscopic disorder (Cohen and Jortner 1983a,b, Stroud and Pan 1979, Shklovskii 1977, Straley, 1980a, b, Bergman *et al* 1983). Exact results are known for the critical behaviour of Hall conductivity λ and Hall coefficient R in two dimensions (Shklovskii 1977, Bergman *et al* 1983). Nagatani (1986) has proposed a fractal model with the 'self-duality' property for determining the critical behaviour of the low-field Hall effect near the two-dimensional percolation threshold. A number of discussions of the critical properties have been given, but there are few reliable calculations of the critical properties in three-dimensional systems.

In this letter we try to determine the critical behaviour of the Hall effect at the three-dimensional percolation threshold with the help of a regular fractal.

In order to determine the critical behaviour of the Hall conductivity near the percolation threshold of an isotropic composite, Bergman *et al* (1983) realised the Hall problem on a two-component discrete lattice as follows: each element of the lattice is a triplet of identical conductors with an ohmic conductance σ_1 or σ_2 that lie along the coordinate axes, and which are electrically unconnected in the absence of a magnetic field H (see figure 1 where a centre of triplets is marked by a circle). In the presence of an H field taken to lie along the z axis, a Hall current will flow through a conductor in the x direction that depends on its Hall conductance (λ_1 or λ_2) and on the voltage across the y conductor of the same triplet. The two types of triplets are placed randomly at all the sites of an FCC lattice, and electrical connections are made at the centres of all the unit cell edges as well as the body-centre points (see figure 1 where an electrical connection is indicated by a full circle). It is easy to see that by making these connections we obtain four simple-cubic, random-bond resistor networks that are electrically unconnected (for $H=0$) but are correlated with each other by virtue of the unconnected triplets used in setting them up. One notes that the basic Hall element (given by the triplet in figure 1) has the feature that a Hall current will flow through a conducting bond only if there is a potential difference along a conducting bond perpendicular to it. This property is essential for a correct representation of a continuous random material of either two or three dimensions.

In order to imitate an infinite cluster in this lattice model of the Hall effect, we must construct a regular fractal model with the following properties. (i) It is composed of triplets (shown by figure 1(*a*)). (ii) It consists of four identical lattices. (iii) The fractal dimensionalities of the fractal and its backbone are very close to those of the

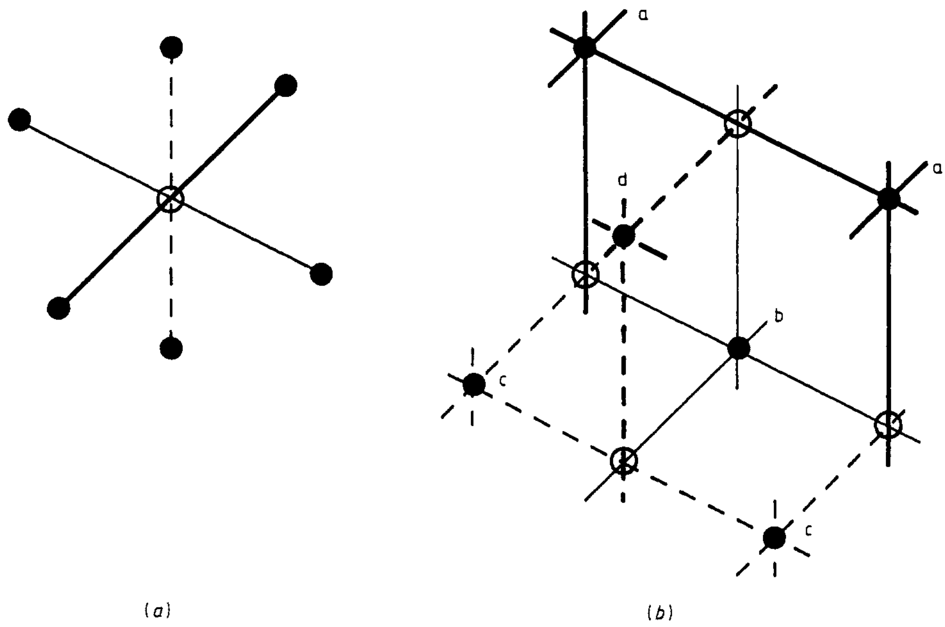

Figure 1. Schematic drawing of a portion of the random-bond resistor networks used to realise the Hall effect in a discrete system in three dimensions. (*a*) A triplet of identical conductors with ohmic and Hall conductances. A triplet is each element of the lattices. (*b*) An FCC lattice of identical but unconnected mutually perpendicular triplets. Electrical connection points that lie on the same connected portion of the network are labelled by identical letters. Thus the 3D network is composed of four unconnected (but correlated) simple-cubic resistor networks (shown by the bold-full, bold-dotted, full and dotted lines). The four networks are electrically unconnected in the absence of a magnetic field, but in the presence of a magnetic field these are correlated.

infinite cluster and its backbone at percolation threshold. The above property (i) is essential for reproducing the correct Hall effect on the discrete lattice. The property (ii) is necessary for the symmetry of an infinite cluster for each axis. The property (iii) is necessary to imitate the geometric texture of an infinite cluster at the threshold. The estimated dimensionalities for the infinite cluster and its backbone are respectively given by $D = 2.5$ and $D_b = 1.74$–2.0 (Stauffer 1985, Hong and Stanley 1983, Herrmann and Stanley 1984).

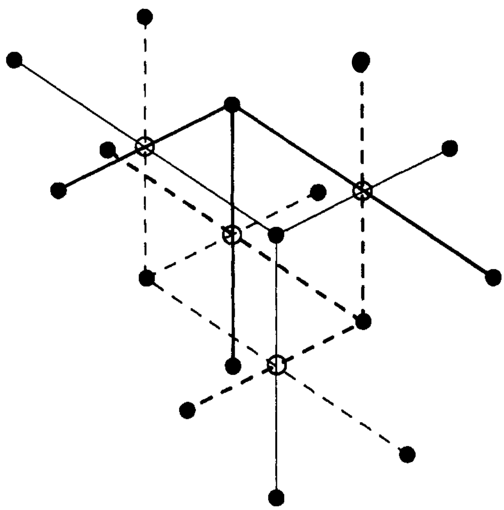

Figure 2. Initiator of the fractal for the infinite cluster. This is composed of four identical lattices (shown by bold-full, bold-dotted, full and dotted lines) with the four triplets.

We construct a fractal with the above properties (i), (ii) and (iii). The initiator and the generator of the fractal are respectively shown by figures 2 and 3(*a*). The initiator is composed of four identical lattices (indicated by bold-full, bold-dotted, full and dotted lines) with the four triplets. The generator is composed of fifteen triplets whose centres are marked by the circles. Its fractal dimension is given by $D = \log 15/\log 3(\sim 2.46)$. The dimensionality of its backbone (shown by figure 3(*b*)) is given by $D_b = \log 8/\log 3(\sim 1.89)$. The fractal, constructed by the initiator and generator (shown by figures 2 and 3(*a*)), thus has the properties (i), (ii) and (iii). The generator of the fractal for its backbone (shown by figure 3(*b*)) is composed of three identical lattices (indicated by bold-full, bold-dotted and full lines). By using a natural decimation procedure, it produces a renormalised triplet, scaled up by a factor $b = 3$. We note that any dangling bonds within the fractal do not contribute to the low-field Hall effect because these contribute to the higher-order terms of a magnetic field H rather than the first-order term.

In the presence of an H field taken to lie along the z axis, we calculate the total ohmic and Hall conductivities between the endpoints in the x direction (or in the y direction), and then derive the exponents describing the power law dependence on scale length L of the conductivities.

In addition to the ohmic conductance σ_a of each member a of the unit element, there is also a Hall conductance λ_a and Hall coefficient R_a, connected by

$$\lambda_a = \sigma_a^2 R_a H \qquad \text{for } a \perp H \qquad 0 \text{ for } a \parallel H. \tag{1}$$

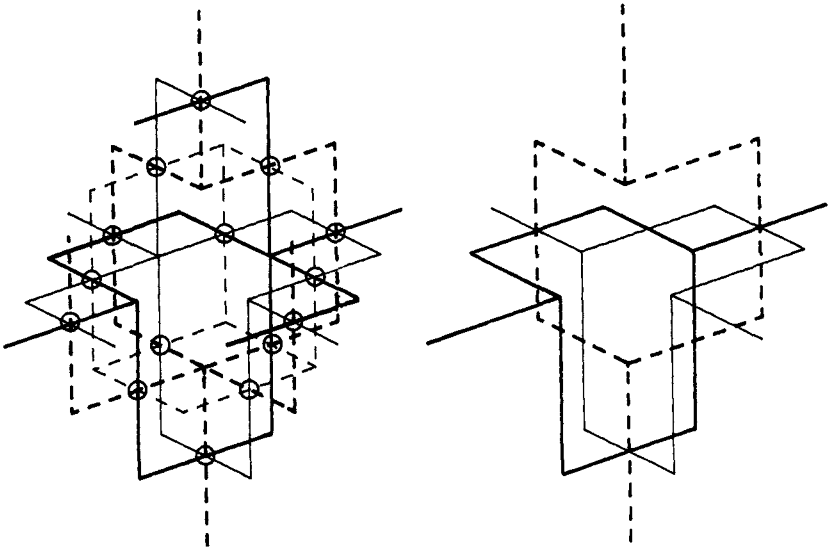

Figure 3. (*a*) Generator of the fractal for the infinite cluster. This is composed of fifteen triplets whose centres are marked by the circles. Its fractal dimension is given by $D = \log 15/\log 3$. The generator consists of the three identical lattices (shown by bold-full, bold-dotted and full lines) and a different lattice (shown by dotted lines). (*b*) Generator of the fractal for its backbone. This is composed of three identical lattices (indicated by bold-full, bold-dotted and full lines).

The current J_a is given by

$$J_a = \sigma_a V_a - \lambda_a V_{a\times H} \tag{2}$$

where $V_{a\times H}$ is the voltage across another conductor of the same unit element—the one that is perpendicular to both a and H. Current conservation at the internal point i leads to the following equation for the potentials V_j:

$$\sum_j \sigma_{ij}(V_i - V_j) + \sum_{ij\times H} \lambda_{ij} V_{ij\times H} = 0 \tag{3}$$

where the first sum over j indicates the summation over the nearest-neighbour sites to i, and the second sum over $ij \times H$ represents the summation over another conductor of the same unit element as the ij bond.

The current, flowing through the renormalised bond in the x direction (or in the y direction), is given by

$$J_x = (2/7)\sigma E_x + (2/7)^2 \lambda E_y + O(\lambda^2) \tag{4}$$

or

$$J_y = (2/7)\sigma E_y - (2/7)^2 \lambda E_x + O(\lambda^2)$$

where we omit the higher-order terms of a magnetic field H other than the first-order term.

The renormalised ohmic and low-field Hall conductances are then given by

$$\sigma' = (2/7)\sigma \tag{5}$$

and

$$\lambda' = (2/7)^2\lambda. \tag{6}$$

The exponents (t/ν and τ/ν), describing the power law dependence on scale length L of the ohmic and Hall conductivities ($L^{-t/\nu}$ and $L^{-\tau/\nu}$), are given by

$$t/\nu = 1 - \log(\sigma'/\sigma)/\log b = 1 + \log(7/2)/\log 3 (\sim 2.14) \tag{7}$$

and

$$\tau/\nu = 1 - \log(\lambda'/\lambda)/\log b = 1 + 2\log(7/2)/\log 3 (\sim 3.28). \tag{8}$$

By the use of the estimated connectedness length exponent $\nu = 0.9$ (Stauffer 1985), the exponents t and τ for the ohmic and Hall conductivities are obtained:

$$t = 1.92 \qquad \text{and} \qquad \tau = 2.95. \tag{9}$$

Our results may be compared with Bergman's estimate: $t = 1.64$ and $\tau = 3.0$ which is calculated by numerical simulation on 3D random-bond resistor networks of size $15 \times 15 \times 15$. The result for t is in good agreement with other more detailed analyses (Stauffer 1985).

In summary, the critical behaviour of the low-field Hall effect in a three-dimensional metal-non-metal composite at the percolation threshold has been determined for the first time by the use of the fractal model.

References

Ben Avraham D and Havlin S 1983 *J. Phys. A: Math. Gen.* **16** L559
Bergman D J, Kantor Y, Stroud D and Webman I 1983 *Phys. Rev. Lett.* **50** 1512
Blumenfeld R and Aharony A 1985 *J. Phys. A: Math. Gen.* **18** L443
Cohen M H and Jortner J 1973a *Phys. Rev. Lett.* **30** 696
—— 1973b *Phys. Rev. Lett.* **30** 699
Deutscher G, Zallen R and Adler J (ed) 1983 *Percolation Structures and Processes* (*Ann. Israel Phys. Soc.*) vol 5 (Bristol: Adam Hilger)
Given J A and Mandelbrot B B 1983 *J. Phys. A: Math. Gen.* **16** L565
Herrmann H J and Stanley H E 1984 *Phys. Rev. Lett.* **53** 1121
Hong D C and Stanley H E 1983 *J. Phys. A: Math. Gen.* **16** L475
Kapitulnik A and Deutscher G 1984 *J. Stat. Phys.* **36** 815
Kirkpatrick S 1979 *Les Houches Summer School on Ill-condensed Matter* vol 32, ed R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland) p 321
Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
—— 1984a *J. Stat. Phys.* **34** 895
—— 1984b *J. Stat. Phys.* **36** 519
Mandelbrot B B and Given J A 1984 *Phys. Rev. Lett.* **52** 1853
Martin J E and Keefer K D 1985 *J. Phys. A: Math. Gen.* **18** L625
Nagatani T 1985a *J. Phys. A: Math. Gen.* **18** L821
—— 1985b *J. Phys. A: Math. Gen.* **18** L1149
—— 1986 *J. Phys. A: Math. Gen.* to be published
Shklovskii B I 1977 *Sov. Phys.-JETP* **45** 152
Stanley H E and Coniglio A 1983 *Ann. Israel Phys. Soc.* **5** 101
Stauffer D 1979 *Phys. Rep.* **54** 1
—— 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Straley J P 1980a *J. Phys. C: Solid State Phys.* **13** 4335
—— 1980b *J. Phys. C: Solid State Phys.* **13** L773
Stroud D and Pan F P 1979 *Phys. Rev.* B **20** 455
Vicsek T 1983 *J. Phys. A: Math. Gen.* **16** L647